Wire Plane Analytic Calculations

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Abstract

Useful formulas for design of TPCs with multiple parallel wire planes are derived using an extension of conformal representation theory previously applied to single-grid ionization chambers.[1] Expressions are given for the electric potential and field lines around the wires, the fraction of electrons collected by the wire planes, and the relation between wire plane voltages and asymptotic fields between planes.

1 Introduction

Conformal representation analysis of 2d electrostatics problems is based on the useful fact that $U(x,y)$ and $V(x,y)$ are both solutions of the Laplace equation if $W = U + iV$ is an analytic function of $z = x + iy$. By convention, V is taken to be the electrostatic potential; contours of constant U are perpendicular to contours of V at all points and trace field lines.

Useful equations for the complex potential W near a plane of regularly spaced wires are given in [1], and are repeated below; for ease of comparison, the choice to define "electric field" to point in the direction of force on electrons has been retained. In the following equations x is perpendicular to the wire plane, d is the center-to-center distance between wires in the plane, r is the radius of the wires, and E_- and E_+ are the asymptotic fields at $x \lesssim -d$ and $x \gtrsim +d$, respectively. A difference between the electric fields $\Delta E = E_+ - E_-$ requires a net electric charge on the wires. The potential due to the wire net charges is very well approximated by

$$W_L = \frac{\Delta E}{2\pi} di \left(\log \sinh \frac{\pi z}{d} - \log \frac{\pi r}{d} \right). \tag{1.1}$$

This potential is zero to $\mathcal{O}\left((\pi r/d)^2/6\right)$ on circles of radius r around $z = 0, \pm d, \pm 2d...$, and gives field $\pm\frac{1}{2}\Delta E$ at large $\pm x$. The average field $\bar{E} = (E_+ + E_-)/2$ induces a dipole moment on the wire plane, whose potential is

$$W_D = -i\bar{E}r^2 \frac{\pi}{d} \coth \frac{\pi z}{d}. \tag{1.2}$$

The corresponding electric field drops to zero at large x. An overall average electric field contributes potential

$$W_E = i\bar{E}z.$$

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2 Multiple Wire Planes

Now we consider the case of three wire planes located at $x = -s, 0, +s$. Let E_0 be the asymptotic field at large negative x, E_1 the asymptotic field between the first and second planes, E_2 be the asymptotic field between the second and third planes, and E_3 the field at large positive x. The potential is

$$\begin{aligned} W &= W_L(z+s;\Delta E_1) + W_D(z+s;\bar{E}_1) \\ &\quad + W_L(z+s;\Delta E_2) + W_D(z+s;\bar{E}_2) \\ &\quad + W_L(z+s;\Delta E_3) + W_D(z+s;\bar{E}_3) \\ &\quad + i\frac{1}{2}(E_0+E_3)z, \end{aligned} \tag{2.1}$$

where $\Delta E_i = E_i - E_{i-1}$ and $\bar{E}_i = \frac{1}{2}(E_i + E_{i-1})$.

Figure 2.1 shows an example of the potentials plotted in a particular case.

The condition for transparency is that no field lines intercept a grid wire; equivalently, there should be no contour at the same potential as the wire extending from the wire, as there is in the top right panel of 2.1. This is entirely determined by the potential near the wire, and the resulting equation is unchanged from that derived in [1]. If electrons are drifting from $-x$ to $+x$, the condition is

$$\frac{E_+}{E_-} > \frac{1+\rho}{1-\rho} \tag{2.2}$$

where $\rho = 2\pi r/d$.

The relation between electric fields and wire plane voltages is somewhat different from that given for the single wire grid in [1]. The potential as a function of x passing through a single wire is

$$V(x) = \frac{\Delta E}{2\pi} d\left(\log\sinh\frac{\pi z}{d} - \log\frac{\pi r}{d}\right) + \bar{E}\left(x - r^2\frac{\pi}{d}\coth\frac{\pi x}{d}\right).$$

For $|x| \gtrsim d$ this is

$$\begin{aligned} V(x) &= \frac{\Delta E}{2}\left(|x| - \frac{d}{\pi}\log\frac{2\pi r}{d}\right) + \bar{E}\left(x - r^2\frac{\pi}{d}\right) + \mathcal{O}\left(e^{-2\pi|x|/d}\right) \\ &= E_+\left(x - \frac{\pi r^2}{d}\right) + \Delta E\frac{d}{2\pi}\left(\frac{r^2\pi^2}{d^2} - \log\frac{2\pi r}{d}\right) \quad (x \gtrsim +d), \\ &= E_-\left(x + \frac{\pi r^2}{d}\right) + \Delta E\frac{d}{2\pi}\left(\frac{r^2\pi^2}{d^2} - \log\frac{2\pi r}{d}\right) \quad (x \lesssim -d), \end{aligned}$$

where E_+ and E_- are the asymptotic fields on the positve x and negative x sides of the wire plane. Defining $t = 2\pi r^2/d = \rho r$ and $l = (r\pi/d)^2 - \log(2\pi r/d) = \rho^2/4 - \log\rho$, the above equation becomes

$$V_\pm(x) = E_\pm(x \mp \frac{1}{2}t) + \Delta E l \quad (\pm x \gtrsim d). \tag{2.3}$$

This is the same as derived in [1], and can be used to find the voltage difference between a solid conducting plate at large negative x and the wire plane at most negative x, or between the wire plane at most positive x and a solid conducting plate at large positive x.

We wish to calculate difference in potential between two adjacent wire planes separated by $\Delta x = s$. Both wires contribution to the potential is zero at their own position. The difference in potential is

$$\Delta V = \left[E_1(s - \frac{1}{2}t_1) + (E_1 - E_0)l_1\right] - \left[E_1(-s + \frac{1}{2}t_2) + (E_2 - E_1)l_2\right], \tag{2.4}$$

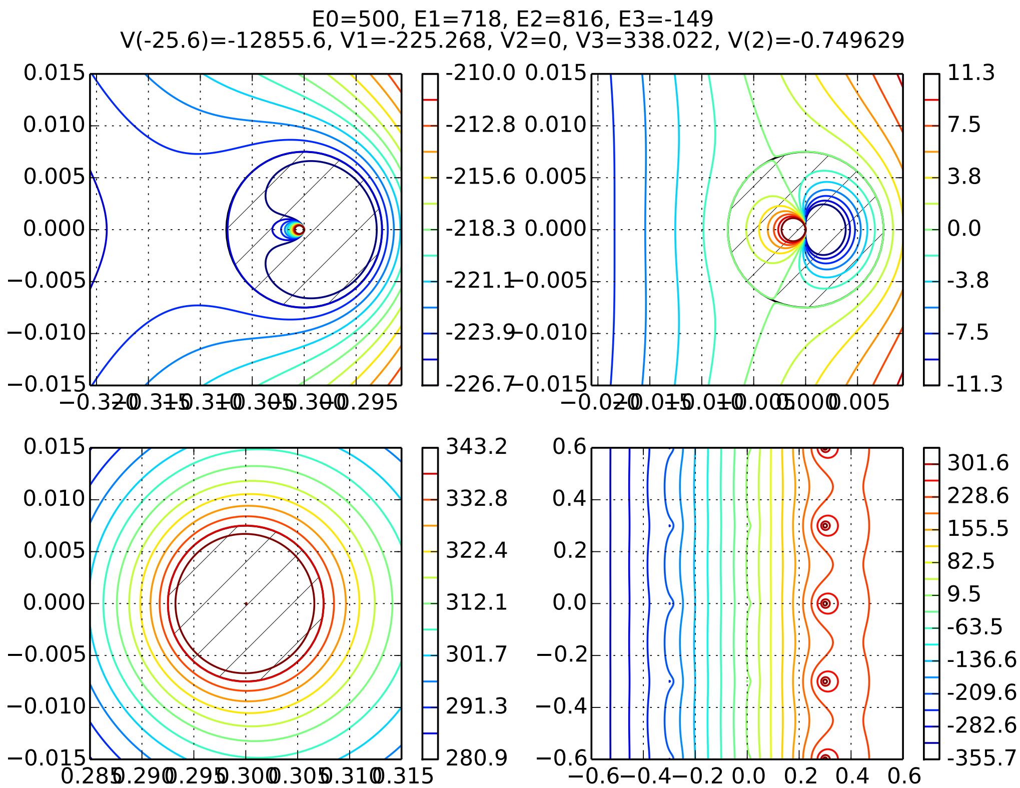

Figure 2.1: Potential plotted for the case where $E_0 = 500$ V/cm, E_1=718 V/cm, $E_2 = 816$ V/cm, $E_3 = -149$ V/cm, $s = d = 0.3$ cm, $r = 75\,\mu$m. Plots show potential near a wire at $x = -s$ (top left), a wire at $x = 0$ (top right), a wire at $x = +s$ (bottom left), and across all three planes (bottom right). Voltages at the wire planes are $V_1, V_2, V_3 = -225\,$V, $0\,$V, $+338$ V, the voltage at -25.6 cm is -12856 V, and the voltage at $+2.0\,$cm is 0 V. Note the saddle point in the potential to the left of the wire in the top left plot: the wire is repulsive to electrons arriving from any direction. In contrast, note the contour at the same potential as the wire intercepting the wire in the top right plot: some field lines arriving from the right terminate at this wire.

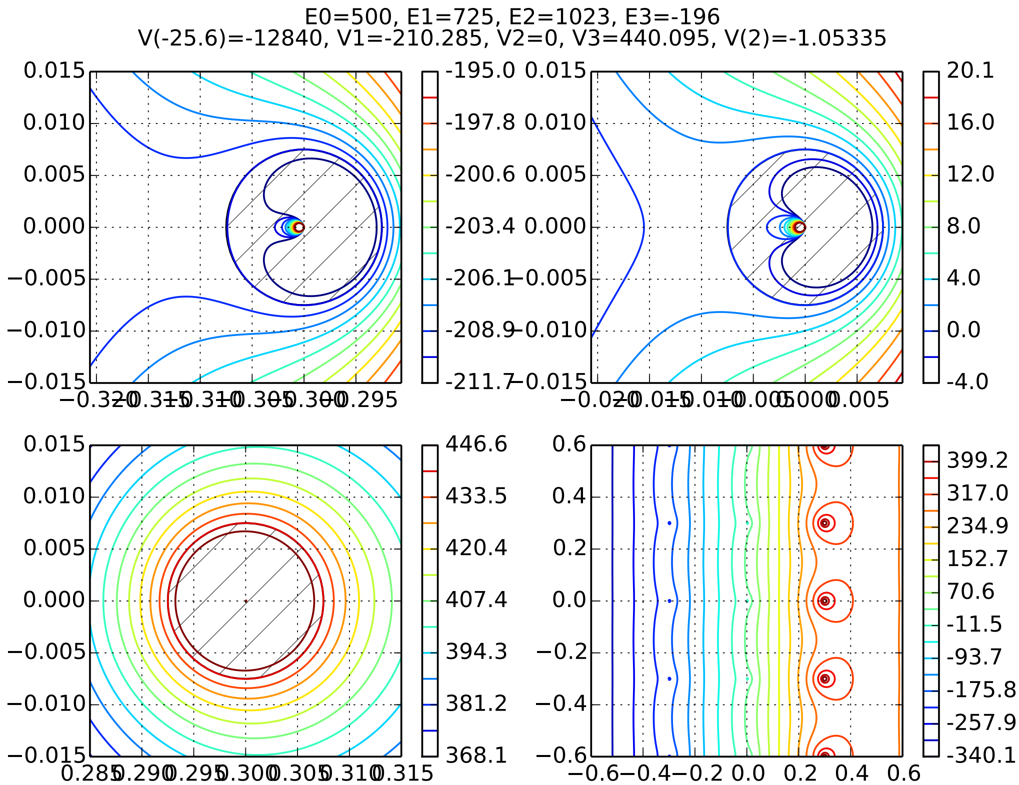

Figure 3.1: First induction plane and collection plane voltages are -210 V and +440 V. The gradient near each induction wire surface is repulsive, and there is complete transparency. The ratio of asymptotic fields is $E_1/E_0 = 1.450$ at the first plane, $E_2/E_1 = 1.411$ at the second.

where t_1, l_1 (t_2, l_2) denote the geometric parameters of the first (second) wire plane, E_1 is the asymptotic field between the two planes, and E_0 (E_2) is the field at large negative (positive) x. In the case where the two planes have identical r and d, this simpifies to

$$\Delta V = E_1(s - t) + (2E_1 - E_0 - E_2)l. \tag{2.5}$$

This equation has been checked numerically against Eq. 2.1.

3 More example plots

Here are some additional figures to demonstrate various cases. In each figure, as in Fig. 2.1, the top left shows a close look at a wire in the first induction plane, top right is a wire in the second induction plane, the bottom left is a collection plane wire, and the bottom right is a view of all three planes. A hash-filled circle to indicates the boundaries of the wires. As expected, there is a constant voltage contour on that surface in each case. All figures have 500 V/cm in the drift region at large negative x, 0 V at the 2nd induction plane, and 0 V at a plane 2 cm behind the collection wires. All have $s = d = 0.3$ cm, $r = 75\,\mu$m, giving a field ratio for grid transparency $(1 + \rho)/(1 - \rho) = 1.373$ (Eq. 2.2). All dimensions are cm, all voltages are volts.

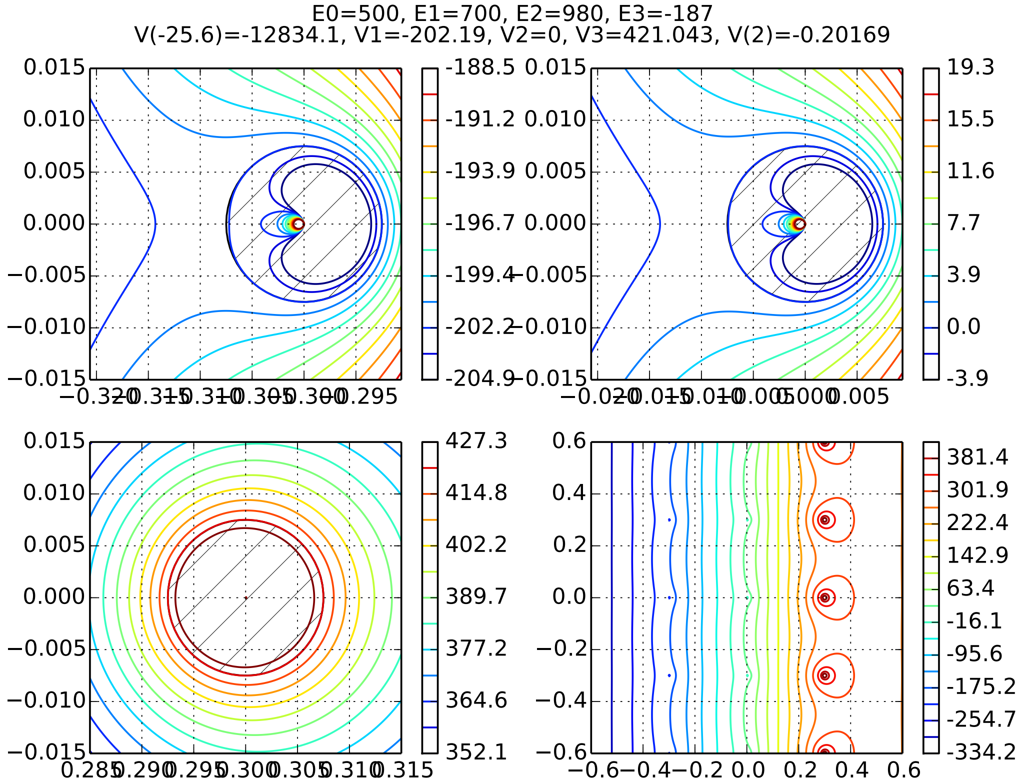

Figure 3.2: Case where both field ratios are 1.400. The corresponding voltages are a little closer to zero in comparison to the -210V, +440V biases case shown in Fig. 3.1. Full transparency is again attained at the induction planes. Note how similar the potentials are around the wires in the two induction planes, as expected since the local potential is determined solely by the asymptotic fields on either side of the wire.

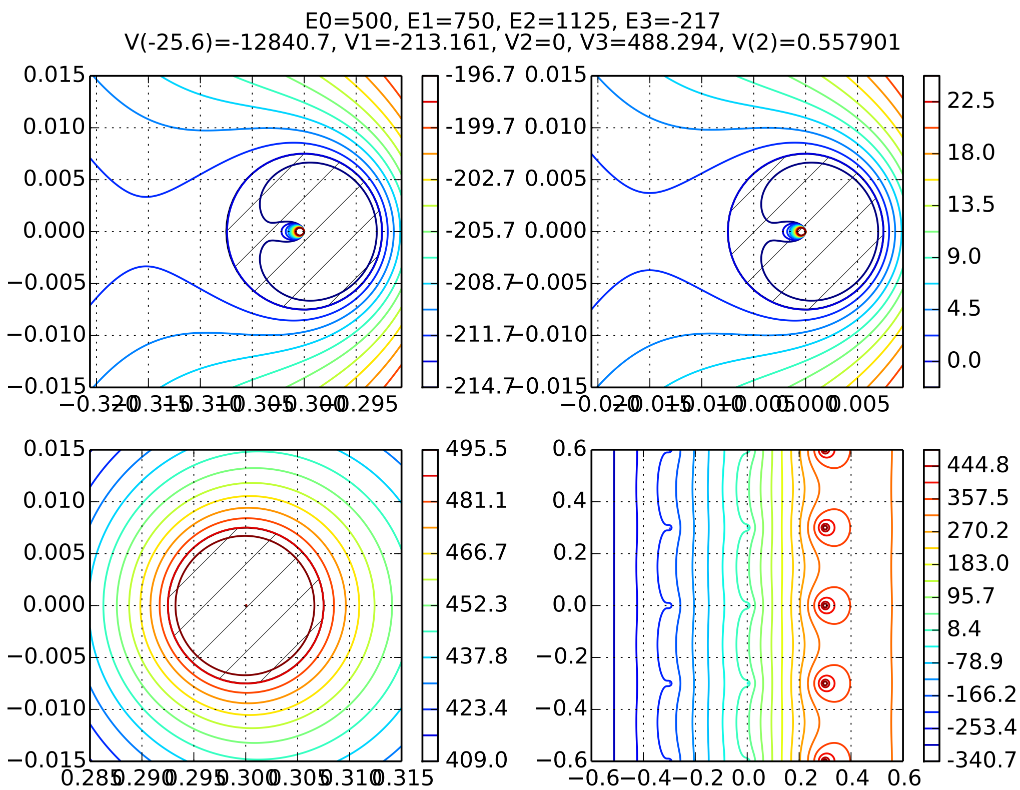

Figure 3.3: Case where both field ratios are 1.500 for comparison.

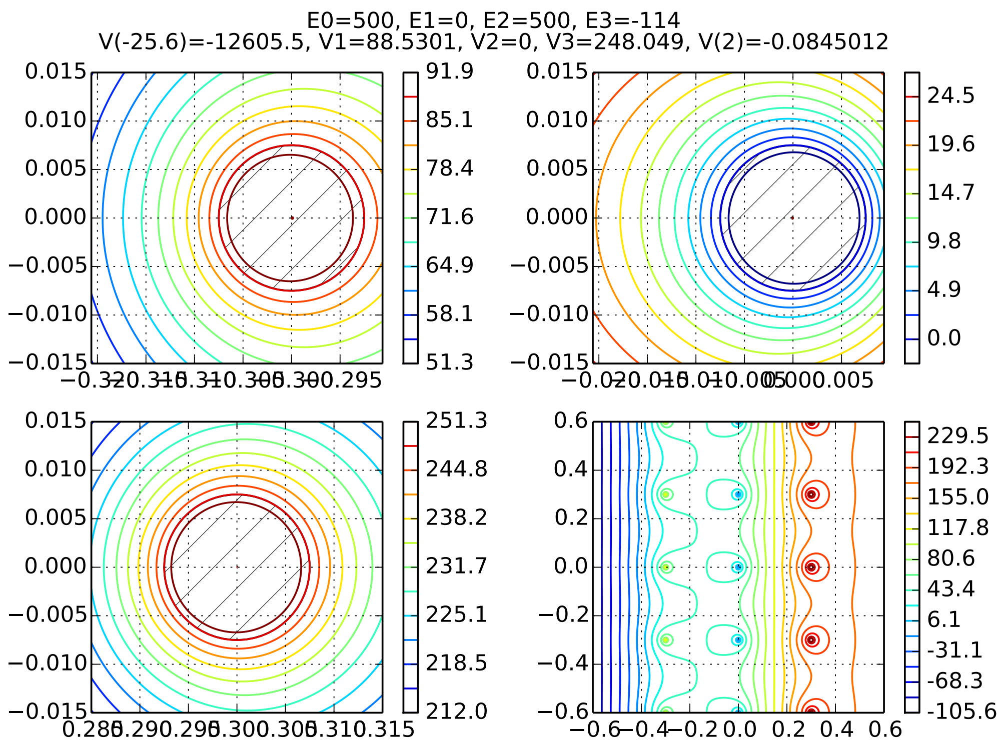

Figure 3.4: Here the voltages have been adjusted to give $E_1 = 0$ in the region between the first and second induction planes, and $E_0 = E_2 = 500$ V/cm. This would not be useful for normal TPC operation, but might be useful if there was reason to pull the first induction plane towards the cathode while pulling the second induction plane in the opposite direction.

4 Source Code

The Python code used to make the figure and check equations is given below.

```
"""WirePlanes3.py -- use conformal mapping to solve for electrostatic
potential in case of three wire planes between two plates, using the
conformal representation technique of Buneman, et al., Can.J.Res.27A
pp. 191-206 (1949). The solution is guaranteed to obey Laplace's
equation by the properties of analytic functions; correctness of the
boundary conditions is checked by verifying potentials on the surfaces
of the wires and plates.

All arguments z = x + iy are coordinates in (x,y) plane, may be numpy arrays.
All functiosn W = U + iV are complex potential, V = electrostatic potential.

:authors: Glenn Horton-Smith, 2015-08-14
"""

import numpy as np
import matplotlib.pyplot as plt

pi = np.pi

def logsinh(z):
        """Protect against overflows in log(sinh(z)).
        The argument z can (should) be a numpy array."""
    z = np.array(z+0.0j, ndmin=1)
    mask = (np.abs(z.real) <= 20.0)
    W = np.zeros_like(z)
    W[mask] = np.log(np.sinh(z[mask]))
    if not mask.all():
        mask = z.real > 20.0
        W[mask] = z[mask]-np.log(2.0)
        mask = z.real < -20.0
        W[mask] = -z[mask]+pi*1j-np.log(2.0)
        W = W.real + 1.j*(np.mod(W.imag+pi,2.0*pi)-pi)
    return W


def W_L(z, dE, d, r):
    """Wire plane potential due to net charge on wires
    for single plane at x=0 and zero potential on the wires.
    z = x+iy = coordinate
    dE = difference in asymptotic field in x direction on two sides of plane,
         dE = E_right - E_left
    d = wire pitch, spacing center to center of wires in this plane
    r = radius of wire
    """
    W = 1j*(dE/(2*pi))*d*(logsinh(pi*z/d)-np.log(pi*r/d))
    return W


def W_D(z, Emean, d, r):
    """Potential due to induced dipole on wires
```

```
    for single plane at x=0 and zero potential on the wires.
    z = x+iy = coordinate
    Emean = average of asymptotic horizontal field on two sides of plane
    d = wire pitch, spacing center to center of wires in this plane
    r = radius of wire
    """
    return -1j*Emean*r**2*(pi/d)*np.tanh(pi*z/d)**-1

def W_E(z, Emean):
    """Potential due to constant mean field
    z = x+iy = coordinate
    Emean = average asymptotic horizontal field
    """
    return 1j*Emean*z

def W_3(z, E0, E1, E2, E3, d, r, s):
    """Combined potential for three wire planes, where
    first wire plane is at x=-s, second at x=0, third at x=+s.
    z = x+iy = coordinate
    E0 = asymptotic field in region x < -s.
    E1 = asymptotic field between first and second wire planes.
    E2 = asymptotic field between second and third wire planes.
    E3 = asymptotic field in region x > +s.
    d = wire pitch, spacing center to center of wires in this plane
    r = radius of wire
    s = wire plane spacing, center to center

    Note that the field between the planes will only approach the
    asymptotic values E1 and E2 in the case of s >> d.  The potential
    returned by this function will be a solution of Laplace equation with
    equipotential surfaces in the correct place regardless.  The E parameters
    can be adjusted as needed to obtain particular wire voltages, or they can
    be adjusted to obtain transparency and voltages found.
    """
    # first calculate the dE and Emeans
    # we're going to superpose fields that are Em+dE/2 on right, Em-dE/2 on left
    #  E0 = Em + 0.5*(-dE1-dE2-dE3)
    #  E1 = Em + 0.5*(+dE1-dE2-dE3)
    #  E2 = Em + 0.5*(+dE1+dE2-dE3)
    #  E3 = Em + 0.5*(+dE1+dE2+dE3)
    #
    #  solution is
    #    Em = (E0+E3)/2
    #    dE1 = E1-E0
    #    dE2 = E2-E1
    #    dE3 = E3-E2
    #
    W1 = W_L(z+s, E1-E0, d, r) + W_D(z+s, 0.5*(E1+E0), d, r)
    W2 = W_L(z,   E2-E1, d, r) + W_D(z,   0.5*(E2+E1), d, r)
    W3 = W_L(z-s, E3-E2, d, r) + W_D(z-s, 0.5*(E3+E2), d, r)
    W = W1 + W2 + W3 + W_E(z, 0.5*(E0+E3))
    return W

def find_surface_potentials(E0, E1, E2, E3, d, r, s, x0, x4):
```

```
    """Checks the surface potentials at the wires and the plates.
    Finds values and also checks uniformity.
    E0, E1, E2, E3, d, r, s have the same meaning as in W_3.
    x0 and x4 are locations far from wires, e.g., a cathode and anode.
    Returns a tuple of two 5-element lists.
    The first element of the tuple contains the voltages [V0, V1, V2, V3, V4],
    where V1,2,3 are the wire voltages and V0,V4 are potential at x0, x4.
    The second element contains the minimum and maximum values found along
    the surfaces.
    """
    V = [0.0]*5
    dV = [0.0]*5
    #-- check planes at two key points
    for i,x in ( (0,x0), (4,x4) ):
        ztest = np.array([x+0j, x+d*0.5j])
        Varr = W_3(ztest, E0, E1, E2, E3, d, r, s).imag
        V[i] = np.average(Varr)
        dV[i] = Varr.max()-Varr.min()
    #-- check planes at four key points
    for i in [1,2,3]:
        x = (i-2)*s
        ztest = np.array([x+r+0j, x+r*1j, x-r+0j, x-r*1j])
        Varr = W_3(ztest, E0, E1, E2, E3, d, r, s).imag
        V[i] = np.average(Varr)
        dV[i] = Varr.max()-Varr.min()
    return np.array(V), np.array(dV)


def plotW_3(rangex, rangey, nsamp, E0, E1, E2, E3, d, r, s):
    """Plot potentials in a range of x and y"""
    #-- make z using numpy.mgrid
    xy = np.mgrid[rangex[0]:rangex[1]:(nsamp*1j),
                  rangey[0]:rangey[1]:(nsamp*1j)]
    z = xy[0]+1j*xy[1]
    W = W_3(z, E0, E1, E2, E3, d, r, s)
    V = W.imag
    Vsurf, dV = find_surface_potentials(E0, E1, E2, E3, d, r, s,
                                        rangex[0], rangex[1])
    V -= Vsurf[2]
    Vsurf -= Vsurf[2]
    print Vsurf,dV
    nwire = 0
    iwire = 0
    for i in range(1,4):
        if rangex[0] < s*(i-2)+r and rangex[1] > s*(i-2)-r:
            iwire = i
            nwire += 1
    if nwire == 0:
        print "no wires in field"
        Vmin = np.min(V)
        Vmax = np.max(V)
        Vlevels = np.linspace(Vmin, Vmax, 20)
    elif nwire > 1:
        print nwire, "wires in field"
```

```
        Vmin = np.min(Vsurf-dV)
        Vmax = np.max(Vsurf+dV)
        Vlevels = np.linspace(Vmin, Vmax, 20)
    else:
        print "1 wire in field"
        Vsliced = V[ np.abs(xy[0]-s*(iwire-2)) > r]
        Vmin = Vsliced.min()
        Vmax = Vsliced.max()
        Vstep = (Vmax-Vmin)/10
        V0 = Vsurf[iwire]
        print Vmin,V0,Vmax
        Vlevels = V0 + np.arange( int((Vmin-V0)/Vstep-1.5),
                                  int((Vmax-V0)/Vstep+2.5) )*Vstep
    print Vlevels
    contour = plt.contour(xy[0], xy[1], V, Vlevels)
    colorbar = plt.colorbar(contour, spacing='proportional',
                            format='%.1f')
    plt.grid()
    if nwire == 1:
        phi = np.linspace(-pi, pi, 60)
        y0 = d*np.round(0.5*(rangey[0]+rangey[1])/s)
        x0 = s*(iwire-2)
        plt.fill(x0+r*np.cos(phi), y0+r*np.sin(phi),
                 fill=False, hatch='/', zorder=-100)
    return (contour, colorbar)


def plot_4pane(E0, E1, E2, E3, d, r, s, x0, x4):
    """Plot a 4-pane view of potentials around wires and an overview of
    all three wire planes.
    """
    V, dV = find_surface_potentials(E0, E1, E2, E3, d, r, s,
                                     x0, x4)
    V -= V[2]
    fig = plt.figure(figsize=[9.0,6.5])
    fig.text(0.5, 0.95,
             "E0=%g, E1=%g, E2=%g, E3=%g" % (E0,E1,E2,E3),
             ha='center', va='bottom')
    fig.text(0.5, 0.95,
             "V(%g)=%g, V1=%g, V2=%g, V3=%g, V(%g)=%g" % (
                 x0, V[0], V[1], V[2], V[3], x4, V[4]),
             ha='center', va='top')
    fig.add_subplot(2,2,1, aspect=1)
    plotW_3([-s-2.75*r,-s+1.25*r], [-2*r,2*r], 400, E0, E1, E2, E3, d, r, s)
    fig.add_subplot(2,2,2, aspect=1)
    plotW_3([-2.75*r,+1.25*r], [-2*r,2*r], 400, E0, E1, E2, E3, d, r, s)
    fig.add_subplot(2,2,3, aspect=1)
    plotW_3([+s-2*r,+s+2*r], [-2*r,2*r], 400, E0, E1, E2, E3, d, r, s)
    fig.add_subplot(2,2,4, aspect=1)
    sd = s+d
    plotW_3([-sd,+sd], [-sd,sd], 400, E0, E1, E2, E3, d, r, s)
    return fig
```

```
def plot_many():
    """Plot potentials for four example cases"""
    figs = []

    rat = 1.5
    fig_rat_1p5 = plot_4pane(500.0, rat*500, rat**2*500, -217.0, 0.3, 75e-4, 0.3, -25.
    fig_rat_1p5.show()
    fig_rat_1p5.savefig('rat_1p5.pdf')
    fig_rat_1p5.savefig('rat_1p5.png')
    figs.append(fig_rat_1p5)

    rat = 1.4
    fig_rat_1p4 = plot_4pane(500.0, rat*500, rat**2*500, -187.0, 0.3, 75e-4, 0.3, -25.
    fig_rat_1p4.show()
    fig_rat_1p4.savefig('rat_1p4.pdf')
    fig_rat_1p4.savefig('rat_1p4.png')
    figs.append(fig_rat_1p4)

    fig_BoVolts = plot_4pane(500.0, 725., 1023., -196., 0.3, 75e-4, 0.3, -25.6, 2.0)
    fig_BoVolts.show()
    fig_BoVolts.savefig('BoVolts.pdf')
    fig_BoVolts.savefig('BoVolts.png')
    figs.append(fig_BoVolts)

    fig_OnePlaneEstVolts = plot_4pane(500.0, 718.0, 816.0, -149.0, 0.3, 75e-4, 0.3, -2
    fig_OnePlaneEstVolts.show()
    fig_OnePlaneEstVolts.savefig('OnePlaneEstVolts.pdf')
    fig_OnePlaneEstVolts.savefig('OnePlaneEstVolts.png')
    figs.append(fig_OnePlaneEstVolts)

    return figs
```

References

[1] Bunemann, et al., "Design of Grid Ionization Chambers", Canadian Journal of Research, 1949, 27a(5): 191-206. doi:10.1139/cjr49a-019.