Path Length of Muons Traversing an Arbitrary Volume

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1 Problem

Deduce the total path length of muons traversing a volume V of arbitrary shape in terms of the rate $dR_H/d\cos\theta$, whose units are number/cm^2/s, of muons incident on a horizontal surface at the location of the volume under consideration, where θ is the polar angle of the muon trajectory with respect to the vertical. You may ignore the energy loss by the muons in the volume and suppose that all muons entering the volume exit it after following a straight-line trajectory within.

2 Solution

The difficulty in accounting for muons that enter or leave the sides of the volume can be avoided by subdividing it into a set of horizontal lamina of vertical thickness h that is small compared to the characteristic length of the volume. Then, to a good approximation every muon that enters a lamina from the top exits it from the bottom. If the muon trajectory makes angle θ to the vertical, then its path length within the lamina is $h/\cos\theta$.

The total path length of muons traversing a lamina of area A in one second is

$$\int_0^1 \frac{Ah}{\cos\theta}\frac{dR_H}{d\cos\theta}\, d\cos\theta = R_H A h \left\langle \frac{1}{\cos\theta} \right\rangle , \tag{1}$$

where

$$R_H = \int_0^1 \frac{dR_H}{d\cos\theta}\, d\cos\theta, \tag{2}$$

and

$$\left\langle \frac{1}{\cos\theta} \right\rangle = \frac{1}{R_H}\int_0^1 \frac{1}{\cos\theta}\frac{dR_H}{d\cos\theta}\, d\cos\theta. \tag{3}$$

Summing over all lamina of the volume, the total path length L per second of muons traversing volume V is

$$L = \sum R_H A h \left\langle \frac{1}{\cos\theta} \right\rangle = R_H V \left\langle \frac{1}{\cos\theta} \right\rangle , \tag{4}$$

noting that $V = \sum Ah$.

Example: At the Earth's surface [1],

$$\frac{dR_H}{d\cos\theta} \approx \begin{cases} 0.06\cos^2\theta/\text{cm}^2/\text{s} & (0 < \theta < \pi/2), \\ 0 & (\pi/2 < \theta < \pi). \end{cases} \tag{5}$$

The total rate of muons incident on a horizontal surface is

$$R_H = \int_0^1 \frac{dR}{d\cos\theta}\, d\cos\theta \approx 0.02/\mathrm{cm}^2/\mathrm{s}, \tag{6}$$

and we can write

$$\frac{dR_H}{d\cos\theta} \approx \begin{cases} 3R_H\cos^2\theta & (0 < \theta < \pi/2), \\ 0 & (\pi/2 < \theta < \pi). \end{cases} \tag{7}$$

The angular average is

$$\left\langle \frac{1}{\cos\theta} \right\rangle = \frac{1}{R_H} \int_0^1 \frac{1}{\cos\theta} \frac{dR_H}{d\cos\theta}\, d\cos\theta = 3 \int_0^1 \frac{\cos^2\theta}{\cos\theta}\, d\cos\theta = \frac{3}{2}\,, \tag{8}$$

such that the average path length of a muon in a horizontal lamina of thickness h is $3h/2$. The total path length L_{total} per second, in cm, in an object of volume V of any shape is, according to eq. (4),

$$L_{\mathrm{total}} = \frac{3}{2} R_H V = 0.03V, \tag{9}$$

for V in cm^3.

For an application of this result, see [2].

A Appendix: Vertical Surface

It is of interest to convert the rate $dR_H/d\cos\theta$ of muons crossing a horizontal surface of unit area to the rate crossing a vertical surface.

For the vertical surface, we adopt a spherical coordinate system with the z' axis perpendicular to the surface (and hence horizontal), with the x' axis horizontal, and the y' axis vertical. Then, muons that cross this surface from the $+z'$ side have a range of θ' from 0 to $\pi/2$, and a range of ϕ' from 0 to π, assuming (as before) that incident muons have only $0 \le \theta \le \pi/2$.

The relations between angles (θ, ϕ) and (θ', ϕ') follow from the components of the unit vector $\hat{\mathbf{r}}$,

$$\hat{r}_x = \sin\theta\cos\phi = \hat{r}_{z'} = \cos\theta', \tag{10}$$

$$\hat{r}_y = \sin\theta\sin\phi = \hat{r}_{x'} = \sin\theta'\cos\phi', \tag{11}$$

$$\hat{r}_z = \cos\theta = \hat{r}_{y'} = \sin\theta'\sin\phi'. \tag{12}$$

We consider muons that pass through a vertical surface element $dA_V = dx'dy'$ from angle $(0 \le \theta' \le \pi/2, 0 \le \phi' \le \pi)$, which corresponds to angle (θ, ϕ) in the original coordinate system. these muons pass through a horizontal area element with $dy = dx'$ and

$$dx = dz' = \frac{\hat{r}_{z'}}{\hat{r}_{y'}} dy' = \frac{\cos\theta'}{\sin\theta'\sin\phi'} dy'. \tag{13}$$

The area dA_H of the horizontal area element that corresponds to the vertical area element dA_V is

$$dA_H = dxdy = \frac{\cos\theta'}{\sin\theta'\sin\phi'}dx'dy' = \frac{\cos\theta'}{\sin\theta'\sin\phi'}\,dA_V. \tag{14}$$

The rate of muons crossing the vertical area element dA_V from solid angle $d\Omega' = d\Omega$ about angles $(\theta',\phi') = (\theta,\phi)$ is

$$dR_V = dA_H\frac{dR_H}{2\pi d\cos\theta}\,d\Omega = dA_V\frac{\cos\theta'}{\sin\theta'\sin\phi'}\frac{dR_H}{2\pi d\cos\theta}\,d\Omega', \tag{15}$$

so that the rate of muons crossing unit vertical surface area from angle (θ',ϕ') is

$$\frac{dR_V}{d\Omega'} = \frac{\cos\theta'}{2\pi\sin\theta'\sin\phi'}\frac{dR_H}{d\cos\theta}\,. \tag{16}$$

The rate R_V of muons crossing a vertical surface of unit area (from one side only) is then

$$R_V = \int\frac{dR_V}{d\Omega'}\,d\Omega' = \int_0^{\pi/2}\sin\theta'\,d\theta'\int_0^{\pi}d\phi'\,\frac{\cos\theta'}{2\pi\sin\theta'\sin\phi'}\frac{dR_H}{d\cos\theta}\,. \tag{17}$$

Example: At the Earth's surface [1],

$$\frac{dR_H}{d\cos\theta} \approx \begin{cases} 3R_H\cos^2\theta & (0<\theta<\pi/2), \\ 0 & (\pi/2<\theta<\pi), \end{cases} \tag{7}$$

where $R_H = 0.02/\mathrm{cm}^2/\mathrm{s}$. Using eqs. (7), (12) and (17) we find

$$\frac{dR_V}{d\Omega'} \approx \begin{cases} \frac{3R_H}{2\pi}\cos\theta'\sin\theta'\sin\phi' & (0<\theta'<\pi/2,\ 0<\phi'<\pi), \\ 0 & (\text{elsewhere}). \end{cases} \tag{18}$$

$$R_V = \frac{3R_H}{2\pi}\int_0^{\pi/2}\cos\theta'\sin^2\theta'\,d\theta'\int_0^{\pi}\sin\phi'\,d\phi' = \frac{R_H}{\pi}\,. \tag{19}$$

B Appendix: Average Path Length

From the rates R_H and R_V of muons incident on horizontal and vertical surfaces, we can calculate the rate of muons entering a volume V with a specified shape. For example, if the volume is a rectangular box of dimensions $l\times w\times h$ where h is the height, then the total rate (assuming that no muons enter the bottom of the box) is

$$R_{\mathrm{total}} = lwR_H + 2(lh+wh)R_V \qquad \text{(box)}. \tag{20}$$

For a right circular cylinder of radius r and height h,,

$$R_{\mathrm{total}} = \pi r^2R_H + 2\pi rhR_V \qquad \text{(cylinder)}. \tag{21}$$

We can now calculate the average path length $\langle L \rangle$ from the total track length (9) and the total rate $R_{\rm total}$ of muons entering the volume,

$$\langle L \rangle = \frac{L}{R_{\rm total}} + \frac{3}{2}\frac{R_H V}{R_{\rm total}} \qquad \text{(cylinder)}. \tag{22}$$

Example: At the Earth's surface, $R_H = 0.02/\text{cm}^2/\text{s}$, and according to eq. (19), $R_V = R_H/\pi$. Then for a cube of edge l,

$$R_{\rm total} = (1 + 4/\pi) l^2 R_H = 2.27 l^2 R_H, \qquad \langle L \rangle = \frac{3l}{2(1 + 4/pi)} = 0.66\, l \qquad \text{(cube)}, \tag{23}$$

and for a cylinder with $d = 2r = h$,

$$R_{\rm total} = 5\pi r^2 R_H, \qquad \langle L \rangle = \frac{3}{5} d = 0.6\, d \qquad \text{(cylinder, } d = h\text{)}. \tag{24}$$

References

[1] T.K. Gaisser *et al.*, *Cosmic Rays*, http://pdg.lbl.gov/2006/reviews/cosmicrayrpp.pdf

[2] S. Palestini and K.T. McDonald, *Space Charge in Ionization Detectors* (March 25, 2007), http://puhep1.princeton.edu/~mcdonald/examples/spacecharge.pdf